In 2015, a town has a population of 100 thousand residents, 60% of whom have college degrees. **SCORE: _____ / 35 PTS**

Every year, 5 thousand people move in to the town, of which 40% have college degrees.

At the same time, 10 thousand people move out of the town (including some who just moved in).

Let $b(t)$ be the number of thousands of residents who have degrees.

Assume that no one earns a degree while in the town,

and that the percentage of people leaving who have a degree is the same as the percentage of people in the town who have a degree.

Find the formula for $b(t)$.

NOTE: Although *b(t)* is not a continuous function, you should solve this problem as if it were.

$$\frac{db}{dt} = 5(0.4) - 10\left(\frac{b}{100-5t}\right) \qquad b(0) = 100(0.6) = 60 \quad 2$$

$$\frac{db}{dt} = \overset{3}{2} - \overset{4}{\frac{2b}{20-t}}$$

$$2 \quad \frac{db}{dt} + \frac{2b}{20-t} = 2$$

$$\mu = e^{\int \frac{2}{20-t}\,dt} \underset{2}{=} e^{-2\ln|20-t|} = \frac{1}{(20-t)^2} \quad 3$$

$$4 \quad \frac{1}{(20-t)^2}\frac{db}{dt} + \frac{2}{(20-t)^3}\,b = \frac{2}{(20-t)^2}$$

$$2 \quad \frac{d}{dt}\frac{1}{(20-t)^2} = \frac{-2(-1)}{(20-t)^3} = \frac{2}{(20-t)^3} \checkmark$$

$$\frac{1}{(20-t)^2}\,b = \int \frac{2}{(20-t)^2}\,dt + C = \overset{3}{\frac{2}{20-t}} + \overset{1}{C}$$

$$3 \quad b = 2(20-t) + C(20-t)^2$$

$$2 \quad 60 = 2(20) + C(20)^2 = 400C + 40$$

$$2 \quad C = \frac{1}{20}$$

$$2 \quad b(t) = 2(20-t) + \frac{1}{20}(20-t)^2$$

Solve the differential equation $(3x^2y - y^3)\,dx + (xy^2 - x^3)\,dy = 0$. **SCORE: _____ / 40 PTS**

5 $y = vx$

$dy = v\,dx + x\,dv$

$(3x^2(vx) - (vx)^3)\,dx + (x(vx)^2 - x^3)(v\,dx + x\,dv) = 0$

3 3 4

$(3vx^3 - v^3x^3)\,dx + (v^2x^3 - x^3)(v\,dx + x\,dv) = 0$

3

3 $2vx^3\,dx + (v^2-1)x^4\,dv = 0$

5 $\int \frac{1}{x}\,dx = \int \frac{1-v^2}{2v}\,dv$

2 $C + \ln|x| = \int (\frac{1}{2}v^{-1} - \frac{1}{2}v)\,dv = \frac{1}{2}\ln|v| - \frac{1}{4}v^2$ 4

2 $Cx = v^{\frac{1}{2}}e^{-\frac{1}{4}v^2}$

4 $Cx = \left(\frac{y}{x}\right)^{\frac{1}{2}}e^{-\frac{1}{4}\left(\frac{y}{x}\right)^2}$

$Cx^2 = \frac{y}{x}e^{-\frac{y^2}{2x^2}}$

2 $Cx^3 = ye^{-\frac{y^2}{2x^2}}$

Solve the differential equation $(3x^2y - y^3)\,dx + (xy^2 - x^3)\,dy = 0$. ALTERNATE SOLUTION

SCORE: _____ / 40 PTS

TRY INTEGRATING FACTOR $\mu = x^a y^b$ (5) TO MAKE DE EXACT

(3) $\underbrace{(3x^{a+2}y^{b+1} - x^a y^{b+3})}_{M}dx + \underbrace{(x^{a+1}y^{b+2} - x^{a+3}y^b)}_{N}dy = 0$

$M_y = 3(b+1)x^{a+2}y^b - (b+3)x^a y^{b+2}$ (3)

$N_x = -(a+3)x^{a+2}y^b + (a+1)x^a y^{b+2}$ (3)

(2) $3(b+1) = -(a+3) \longrightarrow a + 3b = -6$

(2) $-(b+3) = a+1 \longrightarrow a + b = -4$

$2b = -2 \longrightarrow b = -1$ (2)

$a = -3$ (2)

$\mu = x^{-3}y^{-1}$

(2) $\underbrace{(3x^{-1} - x^{-3}y^2)}_{M}dx + \underbrace{(x^{-2}y - y^{-1})}_{N}dy = 0$

(2) $M_y = -2x^{-3}y = N_x$ ✓

$f = \int(3x^{-1} - x^{-3}y^2)\,dx = 3\ln|x| + \frac{1}{2}x^{-2}y^2 + C(y)$ (2) (2)

$f_y = x^{-2}y + C'(y) = x^{-2}y - y^{-1}$ (2)

$C'(y) = -y^{-1}$

$C(y) = -\ln|y|$ (2)

$f = 3\ln|x| + \frac{1}{2}x^{-2}y^2 - \ln|y| = C$ (2)

$\frac{x^3}{y}e^{\frac{y^2}{2x^2}} = C$ (2)

$x^3 e^{\frac{y^2}{2x^2}} = Cy$ (2)

Solve the differential equation $w\,dr + (r - r^2w^4)\,dw = 0$. **SCORE: _____ / 40 PTS**

$$w\frac{dr}{dw} + r - r^2w^4 = 0$$

$$w\frac{dr}{dw} + r = r^2w^4$$

5 $$\frac{dr}{dw} + w^{-1}r = w^3r^2 \quad \text{BERNOULLI}$$

LET $v = r^{-1} \longrightarrow r = v^{-1}$ 5

$$\frac{dr}{dw} = -v^{-2}\frac{dv}{dw}$$

4 $$-v^{-2}\frac{dv}{dw} + w^{-1}v^{-1} = w^3v^{-2}$$ 4

4 $$\frac{dv}{dw} - w^{-1}v = -w^3$$

$$\mu = e^{\int -w^{-1}dw} = e^{-\ln|w|} = \frac{1}{w}$$ 2 2

$$\frac{1}{w}\frac{dv}{dw} - \frac{1}{w^2}v = -w^2$$ 2

2 $$\frac{d}{dw}\frac{1}{w} = -\frac{1}{w^2} \checkmark$$

2 1

$$\frac{1}{w}v = \int -w^2\,dw + C = -\frac{1}{3}w^3 + C$$

$$v = -\frac{1}{3}w^4 + Cw$$ 2

$$r^{-1} = -\frac{1}{3}w^4 + Cw$$ 2

$$r = \frac{1}{-\frac{1}{3}w^4 + Cw} = \frac{3}{Cw - w^4}$$ 2

Find a continuous solution of the initial value problem $(\cos x)\dfrac{dy}{dx} + (\sin x)y = \begin{cases} 1, & x < \frac{\pi}{3} \\ \sin x, & x > \frac{\pi}{3} \end{cases}$, $y(0) = -1$. **SCORE: _____ / 35 PTS**

(2) $\dfrac{dy}{dx} + \dfrac{\sin x}{\cos x} y = \begin{cases} \boxed{\sec x}, & x < \frac{\pi}{3} \\ \boxed{\tan x}, & x > \frac{\pi}{3} \end{cases}$ (2)

$\mu = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln|\cos x|} = \sec x$ (3)

(2)

$(\sec x)\dfrac{dy}{dx} + (\sec x \tan x) y = \begin{cases} \sec^2 x, & x < \frac{\pi}{3} \\ \sec x \tan x, & x > \frac{\pi}{3} \end{cases}$ (5)

(2) $\dfrac{d}{dx} \sec x = \sec x \tan x$ ✓

(2)

$(\sec x) y = \begin{cases} \boxed{\tan x + C_1}, & x < \frac{\pi}{3} \\ \boxed{\sec x + C_2}, & x > \frac{\pi}{3} \end{cases}$ (4)

$y = \begin{cases} \sin x + C_1 \cos x, & x < \frac{\pi}{3} \\ 1 + C_2 \cos x, & x > \frac{\pi}{3} \end{cases}$ (3)

$y(0) = -1 = 0 + C_1 \longrightarrow C_1 = -1$ (2)

AS $x \to \frac{\pi}{3}^-$, $y \to \frac{\sqrt{3}}{2} + (-1)\frac{1}{2} = \frac{\sqrt{3}-1}{2}$ (2)

$x \to \frac{\pi}{3}^+$, $y \to 1 + C_2\left(\frac{1}{2}\right) = \frac{C_2 + 2}{2}$ (2)

$\dfrac{\sqrt{3}-1}{2} = \dfrac{C_2+2}{2} \longrightarrow C_2 = \sqrt{3} - 3$ (2)

(2)

$y = \begin{cases} \sin x - \cos x, & x \le \frac{\pi}{3} \\ 1 + (\sqrt{3}-3)\cos x, & x > \frac{\pi}{3} \end{cases}$